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ON SUBNORMAL OPERATORS

BY

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ABSTRACT. Let T be the adjoint of a subnormal operator defined on a Hilbert space H. For any closed set δ, let $X_T(\delta) = \{x \in H:$ there exists an analytic function $f_x: \mathbf{C}\setminus\delta \to H$ such that $(z - T)f_x(z) \equiv x\}$. It is shown that T is decomposable (resp. normal) if $X_T(\partial G_\alpha)$ is closed (resp. if $X_T(\partial G_\alpha) = \{0\})$ for a certain family $\{G_\alpha\}$ of open sets. Some of the results are extended to the case that T is the adjoint of the restriction of a spectral or decomposable operator to an invariant subspace.

Putnam [17] and Stampfli [20] approach the invariant subspace problem for a hyponormal (cohyponormal) operator T by studying the analytic continuability of the local resolvents $(z - T)^{-1}x$ for individual vectors x in the underlying Hilbert space. Here, by independent proofs, we find some necessary and sufficient conditions for normality or decomposability of a subnormal (cosubnormal) operator in terms of its local resolvents.

1. Preliminaries. Let $B(H)$ denote the algebra of all bounded linear operators defined on a Hilbert space H. We recall the following definitions and facts about the elements of $B(H)$.

(i) An operator $T \in B(H)$ is called spectral if $T = S + Q$ where S is similar to a normal operator, Q is a quasinilpotent operator, and $SQ = QS$ [8, pp. 1939 and 1947]. Moreover T has a (not necessarily orthogonal) resolution of the identity which coincides with that of S.

(ii) The restriction of a normal (resp. spectral) operator to an invariant subspace is called a subnormal (resp. subspectral) operator; the adjoint of a subnormal (resp. subspectral) operator is called a cosubnormal (resp. cosubspectral) operator.

(iii) An operator $T \in B(H)$ is hyponormal if $T^*T - TT^* \geq 0$ and cohyponormal if $T^*T - TT^* \leq 0$.

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(iv) Every subnormal operator is hyponormal.

(v) For an operator $T \in B(H)$ and a closed subset δ of the complex plane $\mathbf{C}$ we define

$$X_T(\delta) = \{x \in H\colon \text{there exists an analytic function } f_x\colon \mathbf{C}\backslash\delta \to H \text{ such that } (z-T)f_x(z) \equiv x\}.$$

The set $X_T(\delta)$ is a hyperinvariant linear manifold of T. If δ and γ are two disjoint closed subsets of $\mathbf{C}$, then

$$X_T(\delta) \cap X_T(\mathbf{C}\backslash\delta^0) = X_T(\partial\delta) \quad \text{and} \quad X_T(\delta \cup \gamma) = X_T(\delta) + X_T(\gamma).$$

(Throughout this paper δ^0 and $\partial\delta$ denote the interior and the boundary of a set δ respectively.) The proof of the latter fact is similar to that of the Riesz decomposition theorem and uses the following identity:

$$(\mu - T)^{-1}f_x(z) = (z-\mu)^{-1}[(\mu - T)^{-1}x - f_x(z)]$$

for $\mu \notin \sigma(T)$.

(vi) An operator $T \in B(H)$ has the single-valued extension property if there exists no nonzero H-valued analytic function f such that $(z-T)f(z) \equiv 0$. If T has the single-valued extension property, so does its restriction to an invariant subspace. If T has the single-valued extension property and $x \in H$ one may define

$$\sigma_T(x) = \bigcap\{\delta\colon x \in X_T(\delta) \text{ and } \delta \text{ closed}\}.$$

It is easy to see that $x \in X_T(\sigma_T(x))$ and $X_T(\delta) = \{x\colon \sigma_T(x) \subseteq \delta\}$.

(vii) An invariant subspace Y of T is called a spectral maximal subspace of T if $Z \subseteq Y$ for all invariant subspaces Z of T such that $\sigma(T|Z) \subseteq \sigma(T\,|\,Y)$. If T has the single-valued extension property and $X_T(\delta)$ is closed, then $X_T(\delta)$ is a spectral maximal subspace of T and $\sigma(T|X_T(\delta)) \subseteq \delta \cap \sigma(T)$ [7, p. 23].

(viii) Let $n \geq 2$ be a positive integer. An operator T is called n-decomposable if for every open covering $G_1, G_2, \ldots, G_n$ of $\sigma(T)$ there exist spectral maximal subspaces $Y_1, Y_2, \ldots, Y_n$ of T such that $H = Y_1 + Y_2 + \cdots + Y_n$ and $\sigma(T\,|\,Y_i) \subseteq \overline{G}_i$ $(i = 1, 2, \ldots, n)$. An operator is called decomposable if it is n-decomposable for all positive integers n [7, p. 57].

(ix) Every normal operator is a spectral operator, and every spectral operator is decomposable. If T is a spectral operator with the resolution of the identity E, then $X_T(\delta) = E(\delta)H$ for all closed sets δ [7, p. 33].

(x) Every n-decomposable operator T has the single-valued extension property and $X_T(\delta)$ is closed for all closed sets δ [14, p. 215] $(n \geq 2)$.

2. Main results. The main purpose of this section is to find some necessary and sufficient conditions for decomposability or normality of a cosubnormal operator (Theorems 1 and 3). Some of the results are extended to co-subspectral operators. Stampfli [20] shows that if T is a hyponormal operator, then $X_T(\delta)$ is closed for all closed sets δ, and if T is cohyponormal, then there exists a closed set δ such that $X_T(\delta) \neq \{0\}$. In this direction we prove the following two lemmas.

Lemma 1. *Let A be a 2-decomposable operator defined on a Hilbert space K. Let H be an invariant subspace of A and let $S = A \mid H$. Then $X_S(\delta)$ is closed and $X_S(\delta) \subseteq H \cap X_A(\delta)$ for all closed sets δ.*

Proof. The fact that $X_S(\delta) \subseteq H \cap X_A(\delta)$ follows from the single-valued extension property for A. Now let x_n be a Cauchy sequence in $x_S(\delta)$ converging to x. Let $A_\delta = A \mid X_A(\delta)$. Since A has the single-valued extension property, it follows that $(\lambda - A_\delta)^{-1}x_n$ has values in H and converges uniformly to $(\lambda - A_\delta)^{-1}x$ on any compact subset of $\mathbf{C}\setminus\delta$. Thus $x \in X_S(\delta)$ and hence $X_S(\delta)$ is closed.

Lemma 2. *Let $N \in B(K)$ be an n-decomposable operator for some $n \geq 2$. Let H be an invariant subspace of N^*. Let $Q\colon K \to K$ be the orthogonal projection onto H and let $T = QNQ \mid H$. Then $QX_N(\delta) \subseteq X_T(\delta)$ for all closed sets δ. Moreover, if $X_T(\overline{\delta}_n)$ and $X_T(\mathbf{C}\setminus\delta_n)$ are closed for a sequence $\{\delta_n\}$ of open sets forming a base for the topology of $\mathbf{C}$, then T is n-decomposable and T^* is 2-decomposable.*

Proof. Let $x \in X_N(\delta)$ and let $N_\delta = N \mid X_N(\delta)$. Since $Q(\lambda - N_\delta)^{-1}x$ is analytic outside δ and $(\lambda - T)Q(\lambda - N_\delta)^{-1}x = x$ for $\lambda \notin \delta$, it follows that $Qx \in X_T(\delta)$ and thus $Qx_N(\delta) \subseteq X_T(\delta)$. Next let $G_1, G_2, \ldots, G_n$ be an open covering of $\sigma(T)$. Let G_{n+1} be an open set such that $\overline{G}_{n+1} \cap \sigma(T) = \emptyset$ and $\sigma(N) \subseteq G_1 \cup G_2 \cup \cdots \cup G_{n+1}$. Let $x \in H$. We have $x = x_1 + x_2 + \cdots + x_n$ with $x_i \in X_N(\overline{G}_i)$, $i = 1, 2, \ldots, n-1$, and $x_n \in X_N(\overline{G_n \cup G_{n+1}})$. Since $X_B(F) = X_B(F \cap \sigma(B))$, it follows that $Qx_i \in X_T(\overline{G}_i)$ $(i = 1, 2, \ldots, n)$ and thus

$$H = \sum_{1 \leq i \leq n} X_T(\overline{G}_i). \tag{†}$$

Now assume $X_T(\overline{\delta}_n)$ and $X_T(\mathbf{C}\setminus\delta_n)$ are closed, where $\{\delta_n\}$ is a sequence of open sets forming a base for the topology of $\mathbf{C}$. We claim T has the single-valued extension property. Assume, if possible, that there exists a nonzero H-valued analytic function f on some disc $|z - z_0| < r$ such that

$(z-T)f(z) \equiv 0$. Let $f(z) = \sum a_n(z-z_0)^n$ and let $z_0 \in \delta_k \subset \bar{\delta}_k \subset \{z: |z-z_0| < r\}$ for some k. Since $M = X_T(\bar{\delta}_k)$ is closed, $f^{(n)}(z) \in M$ for all $z \in \delta_k$ and thus $f(z) \in M$ for $|z-z_0| < r$. Choose z_1 in the unbounded component of $\mathbf{C}\backslash\bar{\delta}_k$ such that $|z_1 - z_0| < r$, and $f(z_1) \neq 0$. It follows that there exists a H-valued analytic function g on $\mathbf{C}\backslash\bar{\delta}_k$ with $(z-T)g(z) \equiv f(z_1)$. On the other hand $(z-z_1)^{-1}f(z_1)$ is a H-valued analytic function defined for $z \neq z_1$ which agrees with $g(z)$ on the unbounded component of $\mathbf{C}\backslash\sigma(T)$. Thus $g(z) = (z-z_1)^{-1}f(z_1)$ for z in the unbounded component of $\mathbf{C}\backslash\bar{\delta}_k$, a contradiction. Hence T has the single-valued extension property.

Let δ be an arbitrary closed set. For each point $z \notin \delta$ there exists an integer $k(z)$ such that $z \in \delta_{k(z)} \subseteq \bar{\delta}_{k(z)} \subseteq \mathbf{C}\backslash\delta$. Since T has the single-valued extension property, it follows that

$$X_T(\delta) = \bigcap_{z \notin \delta} X_T(\mathbf{C}\backslash\delta_{k(z)})$$

and thus $X_T(\delta)$ is closed. Therefore, in view of §1 (vii) and formula (†), T is an n-decomposable operator. The last assertion follows from the fact that the adjoint of a 2-decomposable operator is 2-decomposable [10, p. 1057].

Remark 1. In Lemma 2, let δ be a closed set such that $\sigma(T) \cap \delta^0 \neq \emptyset$. If $\sigma(N) \cap \delta^0 = \emptyset$, then $\delta^0 \subset \sigma_p(T)$ and thus $X_T(\delta) \neq \{0\}$. On the other hand, if $\sigma(N) \cap \delta^0 \neq \emptyset$, then $X_N(\delta) \neq \{0\}$ and thus $QX_N(\delta) \neq \{0\}$ [1, proof of Lemma 1.4]. Hence, again, $X_T(\delta) \neq \{0\}$.

Remark 2. The proof of Lemma 2 suggests the following proposition:

Let T be an operator on some Banach space Y. Let δ_n be a sequence of open sets forming a base for the topology of $\mathbf{C}$. If $X_T(\bar{\delta}_n)$ is closed for all n, then T has the single-valued extension property (*cf.* [2, *Proposition* 1.4]).

The following theorem contains a necessary and sufficient condition for decomposability of a cosubspectral operator.

Theorem 1. *Let $N \in B(K)$ be a spectral operator, and let H, T, and Q be as in Lemma 2. If $X_T(\partial\delta)$ is closed for some closed set δ, then $X_T(\delta)$ and $X_T(\mathbf{C}\backslash\delta^0)$ are closed, and $H = X_T(\delta) + X_T(\mathbf{C}\backslash\delta^0)$. In particular if $X_T(\partial\delta_n)$ is closed for a sequence $\{\delta_n\}$ of open sets forming a base for the topology of $\mathbf{C}$, then T is decomposable and T^* is 2-decomposable.*

Proof. Assume $X_T(\partial\delta)$ is closed. Let x_n be a Cauchy sequence in $X_T(\delta)$ converging to x. Let E be the resolution of the identity for N. Since $QE(\mathbf{C}\backslash\delta)x_n \in X_T(\mathbf{C}\backslash\delta^0)$ and $x_n - QE(\delta)x_n \in X_T(\delta)$ (Lemma 2), it follows that $QE(\mathbf{C}\backslash\delta)x_n \in X_T(\partial\delta)$ and thus $QE(\mathbf{C}\backslash\delta)x \in X_T(\partial\delta)$. Hence x ($= QE(\delta)x$

$+ QE(\mathbf{C}\setminus\delta)x)$ is in $X_T(\delta)$. This shows that $X_T(\delta)$ is closed. By a similar proof $X_T(\mathbf{C}\setminus\delta^0)$ is closed. Since $x = QE(\delta)x + QE(\mathbf{C}\setminus\delta)x$ for all $x \in H$, $H = X_T(\delta) + X_T(\mathbf{C}\setminus\delta^0)$. The rest of the proof follows from Lemma 2.

In the following we write $H = M \oplus N$ if M and N are two (closed) subspaces of H, $M \cap N = \{0\}$, and $H = M + N$.

Lemma 3. *Let $N \in B(K)$ be a spectral operator and let H, T, and Q be as in Lemma 2. Let E be the resolution of the identity for N. Assume $X_T(\partial\delta) = \{0\}$ for some closed set δ. Then $H = X_T(\delta) \oplus X_T(\mathbf{C}\setminus\delta^0)$ and $\|P\| \le L$, where $P: H \to H$ is the projection onto $X_T(\delta)$ parallel to $X_T(\mathbf{C}\setminus\delta^0)$ and $L = \sup\{\|E(\sigma)\| : \sigma$ Borel$\}$.*

Proof. In view of Theorem 1, $X_T(\delta)$ and $X_T(\mathbf{C}\setminus\delta^0)$ are closed, and $H = X_T(\delta) + X_T(\mathbf{C}\setminus\delta^0)$. Since $X_T(\delta) \cap X_T(\mathbf{C}\setminus\delta^0) = X_T(\partial\delta) = \{0\}$, $H = X_T(\delta) \oplus X_T(\mathbf{C}\setminus\delta^0)$. Therefore P is well defined and $Px = QE(\delta)x$. This shows that $\|P\| \le L$. Q.E.D.

If T is a spectral operator on a separable Hilbert space and $\{C_\alpha\}$ is a family of disjoint Jordan curves, then $X_T(C_\alpha) = \{0\}$ for all but a countable number of α. For a cosubspectral operator the following converse is true.

Theorem 2. *Let N, T, K, H and Q be as in Lemma 3. Assume $X_T(\partial\delta_n) = \{0\}$ for a sequence $\{\delta_n\}$ of open sets forming a base for the topology of $\mathbf{C}$. Then T is a spectral operator. Moreover if N has an orthogonal resolution of the identity, so does T.*

Proof. We use a "characterization" of spectral operators stated in Theorem XVI. 4.5 of [8, p. 2147].

Note first that since T is decomposable (Theorem 1), T has the single-valued extension property and $X_T(\delta)$ is closed for all closed sets δ. This proves conditions (A) and (C) of the "characterization".

Now we show that if δ is closed and $E(\delta) = 0$, then $X_T(\delta) = \{0\}$ (E is the resolution of the identity for N). Let $\{\sigma_n\}$ be the subsequence of $\{\delta_n\}$ consisting of all δ_n which lie entirely in $\mathbf{C}\setminus\delta$. Let $\gamma_1 = \sigma_1$ and

$$\gamma_n = \sigma_n \Big\backslash \bigcup_{i<n} \sigma_i \qquad (n = 2, 3, \ldots).$$

Let $x \in X_T(\delta)$. We prove by induction that $QE(\gamma_n)x = 0$ ($n = 1, 2, \ldots$). Since $QE(\gamma_1)x = QE(\sigma_1)x = x - QE(\mathbf{C}\setminus\sigma_1)x \in X_T(\partial\sigma_1)$, $QE(\gamma_1)x = 0$. Assume $QE(\gamma_i)x = 0$ for $i = 1, 2, \ldots, n-1$. It follows that

$$QE(\gamma_n)x = QE(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)x$$

and thus

$$QE(\gamma_n)x \in X_T(\partial\gamma_n \cap \partial(\gamma_1 \cup \cdots \cup \gamma_n)) \subseteq X_T(\partial\sigma_n).$$

Hence $QE(\gamma_n)x = 0$. Therefore $x = QE(\delta)x + \Sigma\, QE(\gamma_n)x = 0$ which implies that $X_T(\delta) = \{0\}$.

Let σ and γ be two disjoint closed sets. There exists a Cauchy domain δ such that (a) $\sigma \subset \delta$, (b) $\gamma \subset \mathbf{C}\backslash\bar{\delta}$, and (c) $E(\partial\delta) = 0$. It follows from Lemma 3 that

$$\|x\| \leq L\|x + y\| \qquad (x \in X_T(\sigma),\ y \in X_T(\gamma)),$$

where $L = \sup\{\|E(\delta)\|: \delta \text{ Borel}\}$. This proves condition (B) of the "characterization".

Let E be as above. Let δ be a closed set and let σ_n be an increasing sequence of closed sets converging to $\mathbf{C}\backslash\delta$. Since

$$x = \lim[QE(\delta)x + QE(\delta_n)x]$$

for all $x \in H$, it follows from Lemma 2 that every closed set δ is in the class $\mathcal{S}_1(T)$ of all sets σ with the property that vectors of the form $x + y$ with $\sigma_T(x) \subseteq \sigma$ and $\sigma_T(Y) \subseteq \mathbf{C}\backslash\sigma$ are dense in H [8, p. 2138]. Therefore to each closed set δ there corresponds a unique projection $F(\delta) \in B(H)$ such that $F(\delta)x = x$ if $\sigma_T(x) \subseteq \delta$ and $F(\delta)x = 0$ if $\sigma_T(x) \subseteq \mathbf{C}\backslash\delta$ [8, p. 2138].

Now let δ and σ_n be as above and assume moreover that $X_T(\partial\delta) = X_T(\partial\sigma_n) = \{0\}$ ($n = 1, 2, \ldots$). Let $x \in H$. By the proof of Lemma 3, $x = \lim y_n$ and $\sigma(y_n) \subseteq (\delta \cup \sigma_n) \cap \sigma_T(x)$, where $y_n = QE(\delta \cup \sigma_n)x$ ($n = 1, 2, \ldots$). Applying the Riesz decomposition theorem to $T \mid X_T(\delta \cup \sigma_n)$ yields $y_n = u_n + v_n$, where $\sigma_T(u_n) \subseteq \delta \cap \sigma_T(x)$ and $\sigma_T(v_n) \subseteq \sigma_n \cap \sigma_T(x)$ ($n = 1, 2, \ldots$). This shows that every closed set δ with $X_T(\partial\delta) = \{0\}$ is in the class $\mathcal{S}_2(T)$ of all sets σ having the property that for every $x \in H$ and every $\epsilon > 0$, there are vectors x_1 and x_2 with $\sigma_T(x_1) \subseteq \sigma_T(x) \cap \sigma$, $\sigma_T(x_2) \subseteq \sigma_T(x) \cap (\mathbf{C}\backslash\sigma)$, and $\|x_1 + x_2 - x\| < \epsilon$.

Let $z_0 \in \mathbf{C}$, $\epsilon > 0$, and let $x \in H$. Let $D_r = \{z: |z - z_0| < r\}$ for $r > 0$. There exists a decreasing sequence $\{r(n)\}$ converging to a number $r(\infty)$ such that $0 < r(\infty) < \epsilon$ and $X_T(\partial D_{r(n)}) = \{0\}$ ($n = 1, 2, \ldots, \infty$). Let $\delta = \bar{D}_{r(\infty)}$, $\sigma_n = \mathbf{C}\backslash D_{r(n)}$, $y_n = QE(\delta \cup \sigma_n)x$, $u_n = F(\delta)y_n$, and let $v_n = F(\sigma_n)y_n$. It follows from the proof of Lemma 3 and the uniqueness of the set function F on $\mathcal{S}_1(T)$ that $y_n = F(\delta \cup \sigma_n)x$ and thus $u_n = F(\delta)x$ and $v_n = F(\sigma_n)x$. (Recall that the restriction of F to $\mathcal{S}_2(T)$ is a spectral measure [8, p. 2140].) Hence

$$x = \lim[F(\delta) + F(\sigma_n)]x$$

which implies that δ is in the class $\mathcal{S}(T)$ of all sets $\sigma \in \mathcal{S}_2(T)$ for which there exist closed sets μ_n and ν_n in $\mathcal{S}_2(T)$ with $\mu_n \subseteq \sigma$, $\nu_n \subseteq \mathbf{C}\backslash\sigma$, $n = 1, 2, \ldots$, and

$$x = \lim[F(\nu_n) + F(\mu_n)]x \qquad (x \in H).$$

Since z_0 and ϵ are arbitrary, it follows that every complex number is interior to a set of arbitrarily small diameter belonging to $\mathcal{S}(T)$. This proves condition (D) of the "characterization" and with it the theorem.

Let F_s ($s \in \mathbf{R}$) be the resolution of the identity for a (bounded) Hermitian operator acting in a separable Hilbert space. There exist a family of Hilbert spaces H_s ($s \in \mathbf{R}$) such that the underlying Hilbert space is unitarily equivalent to $\int_{\mathbf{R}}^{\oplus} H_s \, d\mu(s)$.

Moreover if an operator T commutes with all projections F_s, then T is unitarily equivalent to an operator of the form $\int^{\oplus} T_s \, d\mu(s)$, where $T_s \in B(H_s)$. (For the definitions and properties of direct integrals see [13, pp. 496–503].) Since T is invertible if and only if T_s is invertible a.e. $[d\mu]$, it follows that $(\lambda_n - T_s)^{-1}$ exists a.e. $[d\mu]$ simultaneously for all elements of a sequence $\{\lambda_n\}$ dense in $\mathbf{C}\backslash\sigma(T)$. Thus $\sigma(T_s) \subseteq \sigma(T)$ a.e. $[d\mu]$.

In the following by a Jordan domain we mean an open set enclosed by a rectifiable Jordan curve. Theorem 2 can be sharpened for cosubnormal operators as follows.

Theorem 3. *Let $N \in B(K)$ be a normal operator and let T, H, and Q be as in Lemma 2. Let Δ be a totally ordered set and let $\{D_\alpha\}_{\alpha\in\Delta}$ be a fixed increasing chain of Jordan domains such that $X_T(\partial D_\alpha) = \{0\}$ for all $\alpha \in \Delta$ and the area of the set*

$$C(\Delta_1) = \left(\bigcap_{\beta\notin\Delta_1} \bar{D}_\beta\right) \Big\backslash \left(\bigcup_{\beta\in\Delta_1} D_\beta\right)$$

is zero for any cut Δ_1 in Δ. (A subset Δ_1 of Δ is a cut in Δ if any element in Δ_1 is less than any element in the complement of Δ_1.) Then T is a normal operator.

Proof. Assume without loss of generality that H is separable and that T has no nontrivial reducing invariant subspace on which it is normal. We claim $H = \{0\}$. Let P_α be the projection onto $X_T(\bar{D}_\alpha)$ parallel to $X_T(\mathbf{C}\backslash D_\alpha)$. Since $\|P_\alpha\| \leq 1$, (Lemma 3), $\{P_\alpha\}$ is an increasing sequence of orthogonal projections commuting with T.

Let π be a chain of projections obtained from the completion of $\{P_\alpha\}$. We claim π has no gap. Assume, if possible, (P^-, P^+) is a gap in π. Let

$\Delta_1 = \{\alpha \in \Delta: P_\alpha \le P^-\}$. Then $M = (P^+ - P^-)H$ is a nontrivial reducing invariant subspace of T and $\sigma(T \mid M) \subseteq \sigma(T|(P_\beta - P_\alpha)H) \subseteq \bar{D}_\beta \backslash D_\alpha$ for all $\alpha \in \Delta_1$ and $\beta \in \Delta_1$. Thus the area of $\sigma(T \mid M)$ is zero and hence $T \mid M$ is a normal operator, a contradiction [16]. Therefore there exists a (strictly increasing) resolution of the identity F_s $(0 \le s \le 1)$ (belonging to a Hermitian operator) whose range coincides with π [5, Theorem 18.1]. Thus (up to unitary equivalence):

$$H = \int_{[0,1]}^{\oplus} H_s \, d\mu(s) \quad \text{and} \quad T = \int_{[0,1]}^{\oplus} T_s \, d\mu(s),$$

where T_s is cohyponormal a.e. $[d\mu]$. (Actually Bastian [3] shows that T_s is cosubnormal a.e. $[d\mu]$.)

For $[a, b] \subseteq [0, 1]$ let

$$T_{[a,b]} = \int_{[a,b]}^{\oplus} T_s \, d\mu(s) \quad \text{and} \quad H_{[a,b]} = \int_{[a,b]}^{\oplus} H_s \, d\mu(s).$$

It is easy to see that $H_{[a,b]} = (F_b - F_a)H$ and $T_{[a,b]} = T \mid H_{[a,b]}$. Let $\delta(n, k) = [(k-1)/n, k/n]$ for $k = 1, 2, \ldots, n$, and $n = 1, 2, \ldots$. Since

$$\mu(\{s \in \delta(n, k): \sigma(T_s) \not\subseteq \sigma(T_{\delta(n,k)})\}) = 0$$

for all $\delta(n, k)$, it follows that

$$(*) \qquad \sigma(T_s) \subseteq \bigcap_{(n,k) \in \Gamma(s)} \sigma(T_{\delta(n,k)})$$

a.e. $[d\mu]$, where $\Gamma(s) = \{(n, k): s \in \delta(n, k)\}$. Let s satisfy $(*)$. Let $\Delta_1 = \{\alpha \in \Delta: P_\alpha < F_s\}$, $\Delta_2 = \{\alpha \in \Delta: P_\alpha = F_s\}$, and $\Delta_3 = \Delta \backslash (\Delta_1 \cup \Delta_2)$. Since P_α is constant on Δ_2, it follows that

$$\sigma(T) \subseteq \left(\bigcap_{\alpha \in \Delta_2} \bar{D}_\alpha \right) \cup \left(\bigcap_{\alpha \in \Delta_2} (\mathbf{C} \backslash D_\alpha) \right),$$

and thus $\sigma(T_s) \subseteq C(\Delta_1) \cup C(\Delta_1 \cup \Delta_2)$. Hence the area of $\sigma(T_s) = 0$. This shows that T_s is normal a.e. $[d\mu]$. Therefore T is normal and thus $H = \{0\}$. The proof of the theorem is complete.

Definition. An operator T is said to satisfy a boundedness condition (B) if there exists a positive constant L such that $\|x\| \le L\|x + y\|$ for all $x \in X_T(\delta)$, $y \in X_T(\sigma)$, and all pairs of disjoint closed sets δ and σ. (We do not impose the single-valued extension property on T [8, p. 2138].)

Stampfli [20] shows that a cohynormal operator satisfying a boundedness condition (B) has a nontrivial invariant subspace. The following theorem shows that such cosubnormal (resp. cosubspectral) operators are indeed normal (resp. spectral).

Theorem 4. *A cosubnormal (resp. cosubspectral) operator $T \in B(H)$ satisfying a boundedness condition* (B) *is normal (resp. spectral).*

Proof. Assume without loss of generality that H is separable. Let $N \in B(K)$ be the adjoint of a normal (resp. spectral) extension of T^* and let K be separable. Let E be the resolution of the identity for N. Let $\{C_\alpha\}$ be an arbitrary family of disjoint rectifiable Jordan curves. Since K is separable, $E(C_\alpha) = 0$ for all but a countable number of α. Let δ be a closed set such that $E(\delta) = 0$. Let G_n be a decreasing sequence of open sets converging to δ. The sequence $E(G_n)$ converges strongly to zero as $n \to \infty$. Let $x \in X_T(\delta)$. It follows from the boundedness condition (B) and Lemma 2, that $\|x\| \le L\|x - QE(\mathbf{C}\setminus G_n)x\|$ for all n. Letting $n \to \infty$ yields $x = 0$. Thus $X_T(\delta) = \{0\}$ and hence, in view of Theorem 3 (resp. Theorem 2), T is a normal (resp. spectral) operator.

3. Eigenvalues of cosubnormal operators. Let $\sigma_p(T)$ be the set of all eigenvalues of an operator $T \in B(H)$. Let $\sigma_{p\perp}(T)$ be the set of all eigenvalues λ of T such that the null space $N(\lambda - T)$ reduces T. Let $\sigma_{p0}(T)$ be the set of all complex numbers λ such that λ is in the domain of some nonzero H-valued analytic function $f(z)$ which has a connected domain and satisfies $(z - T)f(z) \equiv 0$. It is true that $\sigma_{p0}(T) \subseteq \sigma_p(T)$ [7, p. 22] and $\sigma_{p0}(T) \cap \sigma_{p\perp}(T) = \emptyset$. (Because if $\lambda \in \sigma_{p\perp}(T)$ and λ is in the domain of an analytic function f satisfying $(z - T)f(z) \equiv 0$, then $f(z) \perp N(\lambda - T)$ for all z and $\lambda \in \sigma_p(T \mid N(\lambda - T)^{\perp})$, a contradiction.) Also if S is the restriction of an operator $N \in B(K)$ to an invariant subspace H (of N), then

$$\sigma(S^*)\setminus\sigma(N^*) \subseteq \sigma_{p0}(S^*). \tag{$**$}$$

(Let Q be the projection onto H and let λ and μ be two points of $\sigma(S^*)$ lying in the same component G of $\mathbf{C}\setminus\sigma(N^*)$. Let x be a nonzero vector in H such that $(\mu - T)x = 0$. Then $f(z) = (z - \mu)^{-1}x - Q(z - N^*)^{-1}x$ $(z \in G\setminus\{\mu\})$ is a nonzero analytic function having λ in its (connected) domain and satisfying $(z - T)f(z) \equiv 0$.) (In view of the Wold decomposition theorem for isometry operators, formula $(**)$ provides another proof for Lemma 1.7 of [7, p. 10].)

The following lemmas study the relation between $\sigma_p(T)$ and the geometrical shape of $\sigma(T)$ for a cosubnormal or cohyponormal operator T.

Lemma 4. *Let T be a cohynormal operator. Let $\lambda \in \partial\sigma(T)$. Assume there exists a constant K and a sequence $\{\lambda_n\}$ in $\mathbf{C}\setminus\sigma(T)$ such that* $\lim \lambda_n = \lambda$ *and* $|\lambda - \lambda_n| \le K \operatorname{dist}(\lambda_n, \sigma(T))$ *for $n = 1, 2, \ldots$. Then $\lambda \in \sigma_{p\perp}(T)$ if $\lambda \in \sigma_p(T)$.*

Proof. Assume without loss of generality that $N(\bar{\lambda} - T^*) = \{0\}$. We claim

$N(\lambda - T) = \{0\}$. By [18, p. 469]

$$\|(\lambda_n - T)^{-1}\| \leq 1/\text{dist}(\lambda_n, \sigma(T)) \leq K/|\lambda - \lambda_n| \quad \text{for } n = 1, 2, \ldots.$$

Therefore $H = N(\lambda - T) \oplus \overline{R}(\lambda - T)$ [12, p. 62]. (Here $\overline{R}$ denotes the closure of the range.) Since $N(\overline{\lambda} - T^*) = \{0\}$, $\overline{R}(\lambda - T) = H$ and thus $N(\lambda - T) = \{0\}$.

(For special cases of Lemma 4 see [15] and [19, p. 135].)

Theorem 5. *Let E be a compact subset of the plane. Let $\mathfrak{A}$ be a family of analytic functions having E in their domains. Let H be the span of $\mathfrak{A}$ in $L^2(E, dxdy)$. Let S be the multiplication by z in H and let $T = S^*$. Then*

(a) $X_s(\delta) = \{0\}$ *for all closed subsets δ of E^0,*

(b) $(E^0)^* \subseteq \sigma_{p0}(T)$,

where $\Delta^* = \{\lambda : \lambda \in \Delta\}$. In particular S and T are not 2-decomposable if $E^0 \neq \emptyset$.

Proof. By the area mean value theorem the elements of H are analytic in E^0. Thus if $f \in X_S(\delta)$, it follows from Lemma 1 that $f(z) = 0$ for all $z \notin \delta$ and thus $f \equiv 0$ on E. This proves (a).

Let λ be the center of a disc $|z - \lambda| < r$ lying entirely on E^0. We can assume without loss of generality that $\lambda = 0$ and $r = 1$. Let V be the bilateral weighted shift $Ve_n = [(n+1)/(n+2)]^{1/2} e_{n+1}$ for $n \geq 0$ and $Ve_n = e_{n+1}$ for $n < 0$ defined on some Hilbert space K_1. Let W be the multiplication by z in $K_2 = L^2(E \setminus D, dxdy)$, where D is the unit disc. Let $K = K_1 \oplus K_2$ and $N = W \oplus V$. It is easy to see that $\sigma(N) \cap D = \emptyset$. In view of [11, Problem 25] the mapping $U: H \to K$ defined by

$$Uf = (f|E \setminus D) \oplus \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} [\pi/(n+1)]^{1/2} e_n$$

is an isometry and $US = NU$. Therefore S is unitarily equivalent to a part of N. Since $D \subseteq \sigma(S)$, it follows from (**) that $D \subseteq \sigma_{p0}(T)$. Statement (b) is proved.

The last assertion follows from the fact that T does not have the single-valued extension property. The proof of the theorem is complete.

For a compact set X and a (positive) measure μ on X, let $C(X)$, $\mathfrak{R}(X)$, $R(X)$, and $R^2(X, d\mu)$ denote the continuous functions on X, the rational functions with poles off X, the uniform closure of $\mathfrak{R}(X)$, and the closure of $\mathfrak{R}(X)$ in $L^2(X, d\mu)$, respectively.

Theorem 6. *Let X be a compact subset of $\mathbf{C}$ such that, for any open disc D, $X \cap D \neq \emptyset$ implies $R(X \cap \overline{D}) \neq C(X \cap \overline{D})$. Then there exists a com-*

pletely nonnormal cosubnormal operator T *such that*

$$\overline{\sigma_p(T)} = \sigma(T) = X.$$

(*An operator is called completely nonnormal, if it has no nonzero reducing invariant subspaces on which it is normal.*)

Proof. In view of Theorem 5, we can assume without loss of generality that $X^0 = \emptyset$. Let $Y = X^*$. Following the argument in [6, p. 242] we can find a sequence $\{\lambda_n\}$ dense in Y and a sequence of Borel probability measures $\{\mu_n\}$ such that

$$(***) \qquad f(\lambda_n) = \int_Y f\, d\mu_n \qquad (f \in R(Y))$$

and $\mu_n(\{\lambda\}) < 1$. By replacing μ_n by $[\mu_n - \mu_n(\{\lambda_n\})]/[1 - \mu_n(\{\lambda_n\})]$, we can assume without loss of generality that $\mu_n(\{\lambda_n\}) = 0$. Let A_n be the multiplication by z in $R^2(Y, d\mu_n)$. It follows from $(***)$ and the Schwarz inequality that the nonzero linear functional $f \to f(\lambda_n)$, $f \in R(Y)$, has a bounded extension to $R^2(Y, d\mu_n)$ $(n = 1, 2, \ldots)$. Therefore the range of $\lambda_n - A_n$ lies in a closed subspace of codimension 1 of $R^2(Y, d\mu_n)$, and hence $\overline{\lambda}_n \in \sigma_p((A_n)^*)$ $(n = 1, 2, \ldots)$. Obviously λ_n is not an eigenvalue of A_n, because $\mu_n(\{\lambda_n\}) = 0$. Thus A_n is a nonnormal subnormal operator. Let B_n be the completely nonnormal part of A_n. It follows that $\lambda_n \in \sigma(B_n) \subseteq \sigma(A_n) \subseteq Y$. Let

$$S = \Sigma \oplus B_n \quad \text{and} \quad T = S^*.$$

The operator T satisfies all the requirements of the theorem.

Remark 3. Brennan [4, pp. 314–315] constructs a Swiss cheese E with the following properties:

(a) the linear functional $f \to f(\lambda)$ $(f \in \mathfrak{R}(E))$ has a bounded extension to $R^2(E, dxdy)$ for almost every point λ in E (such points λ are called bounded point evaluations of $R^2(E, dxdy)$),

(b) whenever two functions in $R^2(E, dxdy)$ coincide on a set of positive area in E, they coincide a.e. $[dxdy]$.

Let E be such a set and let S be the multiplication by z in $R^2(E, dxdy)$. Let $T = S^*$. It follows that $\sigma_{p0}(T) = \sigma_{p\perp}(T) = \emptyset$, and the area of $E^* \backslash \sigma_p(T)$ is zero. (Note that, in view of Lemma 4, there are points in $\sigma(T)$ which are not eigenvalues of T.) Let G_1 and G_2 be two open sets such that

(i) $\sigma(S) \subseteq G_1 \cup G_2$,

(ii) the sets $E \backslash \overline{G}_i$ $(i = 1, 2)$ have positive areas. Let $f_i \in X_T(\overline{G}_i)$ $(i = 1, 2)$. By Lemma 1, $f_i = 0$ on $E \backslash \overline{G}_i$ and thus $f_i = 0$ on E $(i = 1, 2)$. Thus S (and hence T) is not 2-decomposable, though it has a nowhere dense spectrum.

One may raise the following question.

Question 1. Is there a nonnormal 2-decomposable subnormal operator?

In view of Theorem 3, a negative answer to the following question will provide a negative answer to Question 1.

Question 2. Is there a decomposable operator $T \in B(H)$ such that $X_T(C_\alpha) \neq \{0\}$ for an uncountable number of disjoint (piecewise smooth) Jordan curves C_α?

Remark 4. The proof of Theorem 6 contains a negative answer to a question raised by Putnam in [15, p. 282].

Let X be a compact set. A point $x \in X$ is called a peak point of $R(X)$ if there exists a function $f \in R(X)$ such that $f(x) = 1$ and $f(y) < 1$ for all $y \in X\backslash\{x\}$. (Such a function f is said to peak at x.) Let $p(X)$ denote the set of all peak points of $R(X)$. We prove the following theorem.

Theorem 7. *If $T \in B(H)$ is a cosubnormal operator, then $p(\sigma(T)) \cap \sigma_p(T) \subseteq \sigma_{p\perp}(T)$.*

Proof. Let $\lambda \in p(\sigma(T)) \cap \sigma_p(T)$. We may and shall assume without loss of generality that $\lambda = 0$. Let $S = T^*$, A be the minimal normal extension of S, and let E be the resolution of the identity for A. Let x be a unit vector such that $Tx = 0$. We prove $Sx = 0$. Since $(Sy \mid x) = 0$ for all $y \in H$, $(g(A)x \mid x) = (g(S)x \mid x) = g(0)$ for all $g \in \mathfrak{R}(\sigma(S))$. Thus $(g(A)x \mid x) = g(0)$ for all $g \in R(\sigma(S))$. Hence if $f \in R(\sigma(S))$ and f peaks at 0, then $(f^n(A)x \mid x) = 1$ for $n = 1, 2, \ldots$. (Note that $0 \in p(\sigma(S))$.) Therefore by dominated convergence theorem

$$1 = \lim (f^n(A)x|x) = \lim \int f^n \, d\|Ex\|^2$$
$$= \int \lim f^n \, d\|Ex\|^2 = \|E(\{0\})x\|^2.$$

Thus $Ax = 0$ and consequently $Sx = 0$. It follows that $N(S) \supseteq N(T) \supseteq N(S)$ which completes the proof of the theorem.

Note. In view of Proposition 3.6 and Theorem 6.1 of [22, pp. 13 and 45],

$$\sigma_p(T)\backslash\sigma_{p\perp}(T) \subseteq \sigma(T)\Big\backslash \bigcup \partial G_i,$$

where T is a cosubnormal operator and $\{G_i\}$ is the class of all components of $\mathbf{C}\backslash\sigma(T)$.

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